Technical Note

Local energy balance, specific heats and the Oberbeck–Boussinesq approximation

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ABSTRACT

A thermodynamic argument is proposed in order to discuss the most appropriate form of the local energy balance equation within the Oberbeck–Boussinesq approximation. The study is devoted to establish the correct thermodynamic property to be used in order to express the relationship between the change of internal energy and the temperature change. It is noted that, if the fluid is a perfect gas, this property must be identified with the specific heat at constant volume. If the fluid is a liquid, a definitely reliable approximation identifies this thermodynamic property with the specific heat at constant pressure. No explicit pressure work term must be present in the energy balance. The reasoning is extended to the case of fluid saturated porous media.

1. Introduction

The Oberbeck–Boussinesq approximation, so named after the pioneering works by Oberbeck [1] and Boussinesq [2], is the basis of most of the contemporary studies on natural or mixed convection flows. Very interesting historical surveys on the origins of this approximation are available in the recent papers by Zeytounian [3] and Bois [4].

Although the nature of this approximation is very clear and unambiguous with reference to the mass and momentum balance equations for the fluid, the formulation of the approximated energy balance equation is not so definite and univocal. The questions concerning the energy balance equations are the following:

(A) Which is the specific heat involved in the energy balance?
(B) Is there a pressure work term in the energy balance, proportional to the convective derivative of the pressure field?

The textbooks on fluid dynamics and heat transfer generally give clear answers to these questions. The problem, as it will be discussed in Section 3, is that the answers are different.

In a recent technical note [5], the present author carried out a first analysis of the existing formulations of the local energy balance adopted in the Oberbeck–Boussinesq approximation of buoyant flows.

The purpose of this short paper is to extend the analysis performed in Ref. [5] in order to point out the manifold nature of the energy balance formulations, within the framework of the Oberbeck–Boussinesq approximation, available in the literature.

Then, a thermodynamic argument is proposed in order to give answers to questions (A) and (B). In particular, it will be concluded that the answer to question (A) depends on the fluid being a liquid or a gas. For a perfect gas the answer to question (A) is definitely: "the specific heat at constant volume c_v". For a liquid the answer to question (A) is less definite, but sufficiently reliable: "the specific heat at constant pressure c_p". The answer to question (B) is: "no pressure work term appears in the energy balance". In a final section, the analysis of the energy balance is applied to the topic of buoyant flows in fluid saturated porous media.

2. Mass and momentum balance

As is well known, the Oberbeck–Boussinesq approximation implies that the local mass and momentum balance equations be written as

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{Du_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial p_e}{\partial x_i} - (T - T_0)\beta g_i + v\nabla^2 u_i, \tag{2}$$

where the summation over repeated indices is assumed. In Eq. (2), D/Dt is the substantial or convective derivative. In Eqs. (1) and (2), the properties v and β are referred to the temperature T_0. The implicit assumptions behind Eqs. (1) and (2) are that: $p_e = p - \rho_0 g_i x_i$ is the excess over the hydrostatic pressure, and that one considers the density as coincident with the reference value ρ_0 except for the gravitational body force term. For that term, the density ρ is assumed to be a function of the temperature only, thus considering the dependence on the pressure as negligible. The linear equation of state

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Nomenclature

c	thermodynamic property, Eqs. (11) and (13)
c_p, c_v	specific heat at constant pressure, specific heat at constant volume
$\mathscr{D}_{ij}$	strain tensor, Eq. (5)
D/Dt	substantial or convective derivative
e	internal energy per unit mass
g_i	gravitational acceleration
h	enthalpy per unit mass
H	inter-phase heat transfer coefficient
k	thermal conductivity
p	pressure
p_e	excess over the hydrostatic pressure
q_i	heat flux density
t	time
T	temperature
T_0	reference temperature
u_i	velocity
V_i	seepage velocity

x_i	position vector

Greek symbols

β	isobaric coefficient of thermal expansion
κ_T	coefficient of isothermal compressibility
μ	dynamic viscosity
v	kinematic viscosity
ρ	density
ρ_0	reference density
σ_{ij}	stress tensor
φ	porosity
Φ	dissipation function for a porous medium

Superscripts, subscripts

s, f	solid, fluid
m	average property

$$\rho(T) = \rho_0[1 - \beta(T - T_0)] \tag{3}$$

is implicitly invoked in Eq. (2). In Eq. (3), the dependence on T is assumed to be sufficiently weak to be approximated linearly in the surroundings of the reference value T_0.

3. Energy balance

For the energy balance, the formulation of the Oberbeck–Boussinesq approximation is not so definite in the literature. In fact, one may have Chandrasekhar's [6] and White's [7] formulation

$$\rho_0 c_v \frac{DT}{Dt} = k\nabla^2 T + 2\mu \mathscr{D}_{ij}\mathscr{D}_{ij}, \tag{4}$$

where $\mathscr{D}_{ij}$ is the strain tensor

$$\mathscr{D}_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right). \tag{5}$$

The source term in Eq. (4), $2\mu \mathscr{D}_{ij}\mathscr{D}_{ij}$, is the thermal power generated by the viscous dissipation.

One may have the enthalpy formulation [8,9]

$$\rho_0 c_p \frac{DT}{Dt} = k\nabla^2 T + 2\mu \mathscr{D}_{ij}\mathscr{D}_{ij} + \beta T \frac{Dp}{Dt}, \tag{6}$$

where the last term on the right hand side is an additional source term: the pressure work acting on the fluid element.

Finally, one may have Landau–Lifshitz's [10], Bejan's [11] and Kundu–Cohen's [12] formulation

$$\rho_0 c_p \frac{DT}{Dt} = k\nabla^2 T + 2\mu \mathscr{D}_{ij}\mathscr{D}_{ij}. \tag{7}$$

In order to decide on the correct expression of the energy balance, let us write the general non-approximated form of this balance [6,13], i.e. the local version of the First Law of thermodynamics

$$\rho \frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ij}\mathscr{D}_{ij}, \tag{8}$$

where e is the internal energy per unit mass, σ_{ij} is the fluid stress tensor and $q_i = -k\partial T/\partial x_i$ is the heat flux density. The meaning of the terms on the right hand side of Eq. (8) is straightforward: the first term is the incoming heat flux contribution to the energy

change, while the second term is the mechanical work input due to the forces acting on the boundary of the fluid element. The latter term depends only on mechanical quantities, i.e. on the velocity and pressure fields within the fluid domain. On the other hand, the evaluation of the energy change on the left hand side of Eq. (8) implies the use of thermodynamics.

Thermodynamics ensures that $e = e(T, \rho)$ for every single-phase or two-phase stable equilibrium states. In the special case of a perfect gas, it is well known that $e = e(T)$ [14], so that

$$de = c_v dT. \tag{9}$$

In the case of either a liquid or a real gas, one must rely on the Oberbeck–Boussinesq approximation by assuming that an approximate equation of state $\rho = \rho(T)$ can be applied. This implies that the pressure of the fluid does not change appreciably. Since $\rho = \rho(T)$ and since the pair (T, ρ) yields a unique stable equilibrium state, then one concludes that all the thermodynamic properties may be considered as functions of T. This conclusion holds for the internal energy per unit mass, so that a relationship

$$de = cdT \tag{10}$$

can be established. The thermodynamic coefficient c, in general, does not coincide either with c_v or with c_p. In fact, c is the total derivative of the function $e = e(T, \rho(T))$ with respect to T, and not the partial derivative of $e = e(T, \rho)$ with ρ kept constant. As is well known, the latter is the correct thermodynamic definition of c_v [14]. The equation of state $\rho = \rho(T)$ is one regarding a set of stable equilibrium states of the fluid with approximately the same pressure. Then, one has

$$c = \left(\frac{\partial e}{\partial T}\right)_p. \tag{11}$$

Eq. (11) is not the definition of the specific heat at constant pressure c_p. As is well known [14], the latter is defined as

$$c_p = \left(\frac{\partial h}{\partial T}\right)_p, \tag{12}$$

where $h = e + p/\rho$ is the enthalpy per unit mass. Then, one can easily write the following relationship:

$$c = c_p - \frac{p\beta}{\rho}, \tag{13}$$

where the definition of the coefficient of isobaric expansion

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \tag{14}$$

has been used. Then, c is smaller than c_p and differs from c_v, except for the limiting case of a perfect gas. Indeed, in the latter case, one can easily show that the equation of state of the perfect gas and Eq. (13) ensure that $c = c_v$, so that Eqs. (9) and (10) are perfectly consistent.

With reference to a Newtonian fluid and on account of Eq. (1), one may express $\sigma_{ij}\mathscr{D}_{ij} = 2\mu\mathscr{D}_{ij}\mathscr{D}_{ij}$. Then, by employing Eq. (10), by replacing ρ with ρ_0 and by assuming $k = constant$, Eq. (8) can be approximated as

$$\rho_0 c \frac{DT}{Dt} = k\nabla^2 T + 2\mu\mathscr{D}_{ij}\mathscr{D}_{ij}. \tag{15}$$

In Eq. (15), the value of the thermodynamic coefficient c, given by Eq. (13), is referred to the temperature T_0 like any other fluid property involved in the Oberbeck–Boussinesq system of governing equations. Eq. (15) coincides with Eq. (4) only for the limiting case of a perfect gas and differs from Eq. (7) as $c < c_p$.

One can question the extent to which c_p and c_v differ from c in the case of liquids. For water at atmospheric pressure, a precise evaluation of the discrepancies $(c_p - c)/c_p, (c - c_v)/c_v$ and $(c_p - c_v)/c_p$ can be done by means of the data reported in Appendix C of Bejan's textbook [11]. In fact, on account of Eq. (13), one has

$$\frac{c_p - c}{c_p} = \frac{p\beta}{\rho c_p}, \quad \frac{c - c_v}{c_v} = \frac{\rho(c_p - c_v) - p\beta}{\rho c_v} = \frac{\rho(c_p - c_v) - p\beta}{\rho c_p - \rho(c_p - c_v)}. \tag{16}$$

Fig. 1 evidently suggests that the discrepancy $(c - c_v)/c_v$ is rather close to $(c_p - c_v)/c_p$ and that $(c_p - c)/c_p$ is much smaller. Fig. 2 shows that $(c_p - c)/c_p$ is, on average, of order 10^{-5}. As a consequence, for water at atmospheric pressure, the approximation

$$c \cong c_p \tag{17}$$

is a definitely reliable one. In general, it is not easy to find data for the specific heat at constant volume of a liquid. Usually, thermodynamic tables report the values of c_p, while c_v is evaluated from the Mayer relationship [14]

$$c_p - c_v = \frac{\beta^2 T}{\rho \kappa_T}, \tag{18}$$

where κ_T is the coefficient of isothermal compressibility,

$$\kappa_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T. \tag{19}$$

On account of Eqs. (16) and (18) and of the data reported in Refs. [14,15], one may evaluate the discrepancies $(c_p - c)/c_p, (c - c_v)/c_v$ and $(c_p - c_v)/c_p$ for some organic liquids at 25 °C and atmospheric pressure. These data are reported in Table 1. This table suggests again that Eq. (17) is definitely reliable with an error smaller than 0.009%, while the approximation $c \cong c_v$ would lead to an error higher than 35%.

4. Possible pitfalls

In the preceding section, a thermodynamic strategy has been established to determine the most appropriate formulation of the local energy balance equation within the Oberbeck–Boussinesq approximation. The basis of this approach is twofold.

- The convective derivative of the internal energy per unit mass, De/Dt, is evaluated by considering the thermodynamic process undergone by the fluid element. This process can be reliably modeled as an isobaric process.
- The mechanical work input $\sigma_{ij}\mathscr{D}_{ij}$ is evaluated according to the stress–strain relationship for a Newtonian fluid, as well as to the constraint $\partial u_i/\partial x_i = 0$ satisfied by the velocity field.

In the light of these arguments, three possible pitfalls that can be encountered in the determination of the local energy balance equation are described in the following.

4.1. The isochoric process

One could say that the convective derivative De/Dt can be evaluated by assuming that the fluid element is undergoing an isochoric thermodynamic process [6]. Therefore the validity of Eq. (9) is extended, not only to the perfect gases, but to every fluid system. Then one would be led to Eq. (4) instead of Eq. (15). This
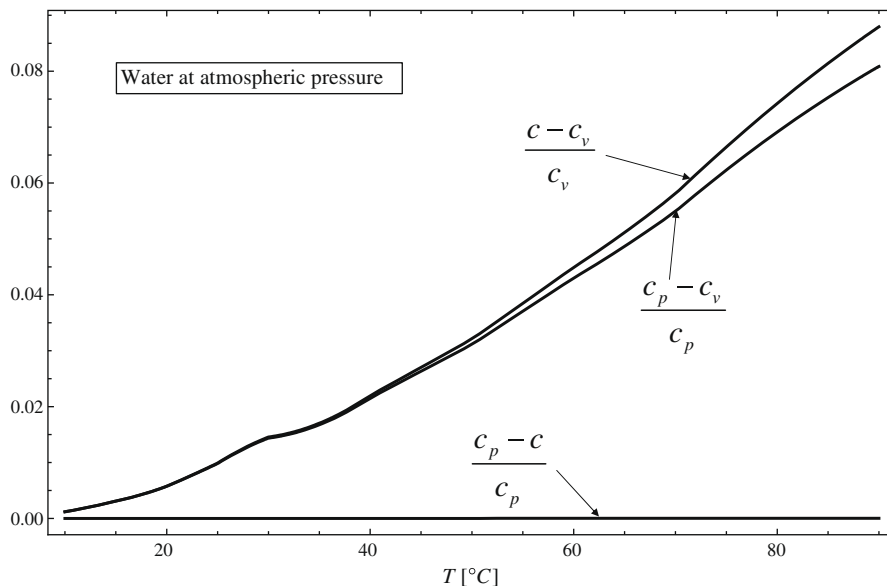

Fig. 1. Comparison of c with c_p and c_v for water at atmospheric pressure.
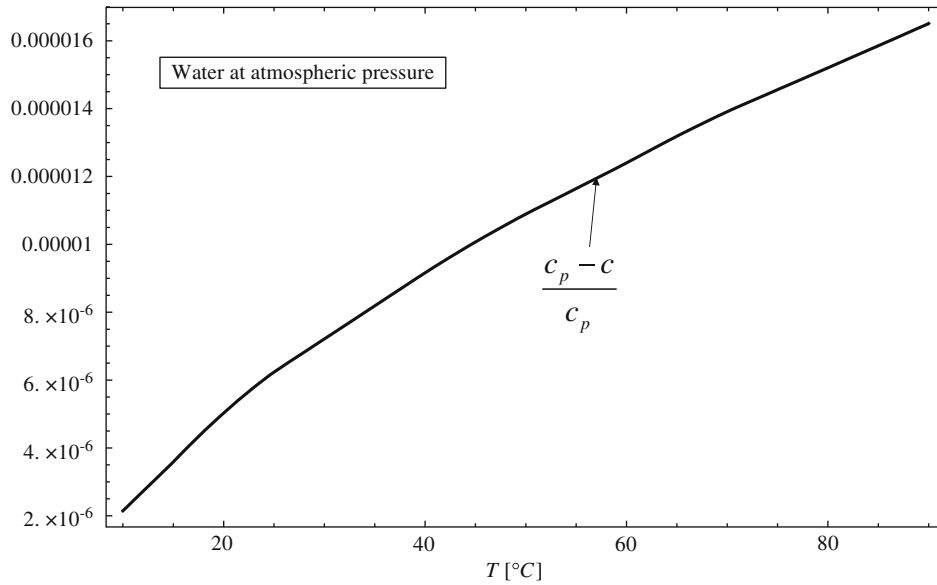
Fig. 2. Comparison of c with c_p for water at atmospheric pressure.

Table 1
Specific heat ratios for some organic liquids at 25 °C and atmospheric pressure.

	$(c_p - c)/c_p$ (%)	$(c - c_v)/c_v$ (%)	$(c_p - c_v)/c_p$ (%)
Acetone (C_3H_6O)	0.0087	40	29
Benzene (C_6H_6)	0.0075	36	26
Ethanol (C_2H_6O)	0.0074	37	27
Methanol (CH_4O)	0.0076	37	27

reasoning is erroneous as the nature of the thermodynamic process undergone by the fluid element is implicitly defined by the equation of state assumed, i.e. Eq. (3). This equation of state is based on the hypothesis that the density may change only as a consequence of temperature variations. In other words, one is assuming that the set of stable equilibrium states available to the fluid element are, with a very good approximation, at constant pressure.

4.2. The expansion–contraction work

A possible misleading argument in the deduction of Eq. (15) is connected to the mechanical work term $\sigma_{ij}\mathscr{D}_{ij}$. This term, as it is clearly explained in Chandrasekhar [6], must be simplified according to the constraint satisfied by the velocity field, i.e. Eq. (1). On the other hand, in some textbooks (see, for instance, Kundu and Cohen [12]), a part of the mechanical work term $\sigma_{ij}\mathscr{D}_{ij}$, namely the expansion–contraction work contribution, $-p\partial u_i/\partial x_i$, is rewritten by forgetting Eq. (1) and by using the exact local mass balance instead,

$$-p\frac{\partial u_i}{\partial x_i} = \frac{p}{\rho}\frac{D\rho}{Dt}. \tag{20}$$

If one brings this term to left hand side of the local energy balance Eq. (8), then one has to evaluate

$$\rho\frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} \tag{21}$$

instead of $\rho De/Dt$. Therefore, by following the thermodynamic argument described in Section 3, one would have

$$\rho de - \frac{p}{\rho}d\rho = \rho c dT + p\beta dT = \rho\left(c_p - \frac{p\beta}{\rho}\right)dT + p\beta dT = \rho c_p dT, \tag{22}$$

where Eqs. (10), (13) and (14) have been used. On account of Eqs. (20)–(22), one would be led to the formulation of the local energy balance expressed by Eq. (7), instead of Eq. (15). However, this is a tricky procedure, since the velocity field evaluated through the Oberbeck–Boussinesq model is constrained to be solenoidal by Eq. (1). Then, even if it is based on an exact mass balance, Eq. (20) cannot be coherently invoked within the Oberbeck–Boussinesq approximation.

4.3. The pressure work

One could evaluate the differential of the internal energy per unit mass, de, by using the definition of enthalpy per unit mass, $h = e + p/\rho$. Then, one has

$$\rho de = \rho dh - dp + \frac{p}{\rho}d\rho. \tag{23}$$

If one assumes that the thermodynamic process undergone by the fluid element is isobaric, then one has

$$dp = 0, \quad dh = c_p dT, \quad d\rho = -\rho\beta dT. \tag{24}$$

Then, Eq. (23) yields

$$\rho de = \rho c_p dT - p\beta dT = \rho\left(c_p - \frac{p\beta}{\rho}\right)dT = \rho c dT, \tag{25}$$

where Eq. (13) has been used. Eq. (25) leads directly to Eq. (15).

If one assumes that the thermodynamic process undergone by the fluid element is isochoric, then one has

$$d\rho = 0, \quad dh = c_p dT + \left(\frac{\partial h}{\partial p}\right)_T dp. \tag{26}$$

The thermodynamic identity

$$\rho\left(\frac{\partial h}{\partial p}\right)_T = 1 - \beta T \tag{27}$$

can be easily proved on the basis of the elementary thermodynamic differential relationships [14]. The complete proof can be found, for instance, in Ref. [13]. Therefore, by substituting Eqs. (26) and (27) into Eq. (23), one obtains

$$\rho de = \rho c_p dT + (1 - \beta T)dp - dp = \rho c_p dT - \beta T dp. \tag{28}$$

From Eq. (28), one justifies the relationship

$$\rho \frac{De}{Dt} = \rho c_p \frac{DT}{Dt} - \beta T \frac{Dp}{Dt}. \tag{29}$$

Obviously, Eq. (29) leads to Eq. (6).

While the procedure based on Eqs. (23)–(25) and leading to Eq. (15) is correct, the procedure based on Eqs. (23), (26)–(29) and leading to Eq. (6) is completely unjustified for the following two reasons.

- The process undergone by the fluid element cannot be considered as isochoric as already pointed out in Section 4.1.
- If one has to suppose that the process is isochoric, the straightforward reasoning is to assume the general validity of Eq. (9), so that one is led to Eq. (4), as described in Section 4.1.

In other words, the misleading thermodynamic analysis behind Eq. (6) adds an unjustified assumption (an isochoric process) to a tricky procedure that replaces the simple conclusion

$$\rho \frac{De}{Dt} = \rho c_v \frac{DT}{Dt} \tag{30}$$

with the more complicated one expressed by Eq. (29). This circumstance induces some reflections on how the erroneous assumption of isochoric process can lead to ambiguous results, depending on the procedure followed. On the contrary, the assumption of isobaric process leads exactly to the same result, *i.e.* Eq. (15), either if the reasoning stems from the evaluation of *de* or, as in the present subsection, if one works out the differential of the enthalpy per unit mass, *dh*.

5. On porous media

The remarks on the appropriate form of the local energy balance in the framework of the Oberbeck–Boussinesq approximation can be easily reformulated with reference to the theory of fluid saturated porous media. In fact, following Nield and Bejan [16], the volume-averaged energy equations for the solid and fluid phases can be written as

$$(1 - \varphi)(\rho c_v)_s \frac{\partial T_s}{\partial t} = (1 - \varphi)k_s \nabla^2 T_s + H(T_f - T_s), \tag{31}$$

$$\varphi(\rho c)_f \frac{\partial T_f}{\partial t} + (\rho c)_f V_i \frac{\partial T_f}{\partial x_i} = \varphi k_f \nabla^2 T_f + \mu \Phi + H(T_s - T_f), \tag{32}$$

where $\mu \Phi$ is the viscous dissipation term obtained through a volume average of the term $2\mu \mathscr{D}_{ij}\mathscr{D}_{ij}$ appearing in Eq. (15). As is well known [17,18], the specific form of the term $\mu \Phi$ depends on the momentum balance model adopted for the fluid saturated porous medium. The inter-phase heat transfer coefficient H in Eqs. (31) and (32) describes the thermal energy flow between the solid and the fluid phase. Since reference is made to the Oberbeck–Boussinesq approximation, all the solid and fluid properties in Eqs. (31) and (32) are evaluated at the reference temperature T_0. With respect to Nield and Bejan [16], Eq. (32) has been adapted on the basis of Eq. (15) in order to include the thermodynamic coefficient c defined through Eq. (13). In cases of local thermal equilibrium between the solid phase and the fluid phase, $T_s = T_f = T$, one can add Eqs. (31) and (32), so that one has

$$(\rho c)_m \frac{\partial T}{\partial t} + (\rho c)_f V_i \frac{\partial T}{\partial x_i} = k_m \nabla^2 T + \mu \Phi, \tag{33}$$

where

$$(\rho c)_m = (1 - \varphi)(\rho c_v)_s + \varphi(\rho c)_f,$$
$$k_m = (1 - \varphi)k_s + \varphi k_f. \tag{34}$$

6. Conclusions

The nature of the Oberbeck–Boussinesq approximation has been revisited in order to clarify some thermodynamic aspects connected to the formulation of the energy balance. The present analysis, motivated by the manifold formulation of this balance in the existing literature, leads to the following conclusions:

1. The energy balance for a fluid, within the Oberbeck–Boussinesq approximation, is given by the equation

   $$\rho_0 c \frac{DT}{Dt} = k\nabla^2 T + 2\mu \mathscr{D}_{ij}\mathscr{D}_{ij},$$

   where $c = c_p - p\beta/\rho$;
2. The thermodynamic coefficient c coincides with the specific heat at constant volume, c_v, for a perfect gas;
3. The thermodynamic coefficient c is definitely well approximated by the specific heat at constant pressure, c_p, for a liquid;
4. No pressure work term of the type $\beta T Dp/Dt$ must be introduced on the right hand side of the energy balance equation.

Finally, the most appropriate form of the energy balance for buoyant flows in a fluid saturated porous medium has been discussed.

Acknowledgment

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